Banzhaf Power

Lecture 13 Section 2.2

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- Coalitions
- Critical Players
- The Banzhaf Power Index
- Examples
- Assignment

Outline

- Coalitions
- 2 Critical Players
- The Banzhaf Power Index
- 4 Examples
- 6 Assignment

Definition (Coalition)

A coalition is a group of players who agree to vote as a block. A winning coalition is a coalition whose votes add up to at least the quota. A losing coalition is a coalition whose votes add up to less than the quota.

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- 5 players?
- In general, if there are n players, then there are $2^n 1$ possible coalitions.
- If we count the empty set (the coalition with nobody in it), then there are 2^n .

Listing Coalitions

- The best way to list the possible coalitions is by size.
 - Start with the empty set (or skip it).
 - Consider all coalitions of a single player: A, B, C, ...
 - Then consider all coalitions of two players by adding a player to the singleton coalitions: AB, AC, BC, ...
 - Then coalitions of three players, then four players, and so on.

By Size



The null set

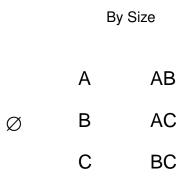
By Size

A

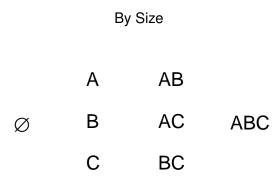
ø B

C

Coalitions of size 1



Coalitions of size 2



Coalition of size 3

Listing Coalitions By "Choice"

- Half of the coalitions include A and half do not include A.
- Half of those that include A also include B and half do not. The same for the half that do not include A.
- Etc.

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Critical Players

Definition (Critical Player)

A critical player of a coalition is a player whose membership in that coalition takes it from a losing coalition to a winning coalition. That is, it is a winning coalition, but if he left it, then it would be a losing coalition.

Example (Coalitions)

Consider the voting system [4:3,2,1]. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
{ <i>A</i> }		
{ <i>B</i> }		
{ <i>C</i> }		
{ <i>A</i> , <i>B</i> }		
{ <i>A</i> , <i>C</i> }		
<i>{B, C}</i>		
{ <i>A</i> , <i>B</i> , <i>C</i> }		

Example (Coalitions)

Consider the voting system [4:3,2,1]. Make a table of all possible coalitions and their critical players.

Coalition	Weight	Critical Players
{ <i>A</i> }	3	
{ <i>B</i> }	2	
{ <i>C</i> }	1	
{ <i>A</i> , <i>B</i> }	5	
{ <i>A</i> , <i>C</i> }	4	
{ <i>B</i> , <i>C</i> }	3	
<i>{A, B, C}</i>	6	

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{ <i>C</i> }	1	
{ <i>A</i> , <i>B</i> }	5	A, B
<i>{A, C}</i>	4	A, C
{ <i>B</i> , <i>C</i> }	3	
<i>{A, B, C}</i>	6	A

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{ <i>B</i> }	2	
<i>{C</i> }	1	
{ <i>A</i> , <i>B</i> }	5	A, B
{ <i>A</i> , <i>C</i> }	4	A, C
<i>{B, C}</i>	3	
{ <i>A</i> , <i>B</i> , <i>C</i> }	6	Α

Notice that A has veto power, but A is not a dictator.

Example (Coalitions)

What if the quota were lowered to 3?

Coalition	Weight	Critical Players
{ <i>A</i> }		
{ <i>B</i> }		
<i>{C}</i>		
{ <i>A</i> , <i>B</i> }		
<i>{A, C}</i>		
<i>{B, C}</i>		
{ <i>A</i> , <i>B</i> , <i>C</i> }		

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<i>{A, B, C}</i>	6	

Example (Coalitions)

What if the quota were lowered to 3?

Coalition	Weight	Critical Players
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{ <i>C</i> }	1	
{ <i>A</i> , <i>B</i> }	5	Α
<i>{A, C}</i>	4	Α
{ <i>B</i> , <i>C</i> }	3	B, C
<i>{A, B, C}</i>	6	

Notice that A is a dictator, but A does not have veto power.

Example (Coalitions)

Consider the voting system [11:9,8,3,1].

		Critical
Coalition	Weight	Players
{ <i>A</i> }		
{ <i>B</i> }		
{ <i>C</i> }		
{ <i>D</i> }		
{ <i>A</i> , <i>B</i> }		
{ <i>A</i> , <i>C</i> }		
{ <i>A</i> , <i>D</i> }		
{ <i>B</i> , <i>C</i> }		
₹ <i>B</i> , <i>D</i> }		
$\{C,D\}$		
{A, B, C}		
$\{A,B,D\}$		
$\{A,C,D\}$		
{B, C, D}		
$\{A, B, C, D\}$		

Example (Coalitions)

Consider the voting system [11:9,8,3,1].

		Critical
Coalition	Weight	Players
{ <i>A</i> }	9	
{ <i>B</i> }	8	
{ <i>C</i> }	3	
{ <i>D</i> }	1	
{ <i>A</i> , <i>B</i> }	17	
{ <i>A</i> , <i>C</i> }	12	
{ <i>A</i> , <i>D</i> }	10	
{ <i>B</i> , <i>C</i> }	11	
{ <i>B</i> , <i>D</i> }	9	
{ <i>C</i> , <i>D</i> }	4	
{ <i>A</i> , <i>B</i> , <i>C</i> }	20	
$\{A, B, D\}$	18	
$\{A,C,D\}$	13	
$\{B,C,D\}$	12	
$\{A,B,C,D\}$	21	

Example (Coalitions)

Consider the voting system [11:9,8,3,1].

		Critical
Coalition	Weight	Players
{ <i>A</i> }	9	
{ <i>B</i> }	8	
{ <i>C</i> }	3	
{ <i>D</i> }	1	
{A, B}	17	A, B
{ <i>A</i> , <i>C</i> }	12	A, C
{ <i>A</i> , <i>D</i> }	10	
{ <i>B</i> , <i>C</i> }	11	B, C
{ <i>B</i> , <i>D</i> }	9	
{ <i>C</i> , <i>D</i> }	4	
$\{A,B,C\}$	20	(none)
$\{A,B,D\}$	18	AB
$\{A,C,D\}$	13	AC
$\{B,C,D\}$	12	BC
$\{A, B, C, D\}$	21	(none)

Example (Coalitions)

Consider the voting system [11:9,8,3,1].

Coalition	Weight	Critical Players
{ <i>A</i> }	9	
₹ <i>B</i> }	8	
{ C }	3	
{ <i>D</i> }	1	
$\{A,B\}$	17	<i>A</i> , <i>B</i>
{ <i>A</i> , <i>C</i> }	12	<i>A</i> , <i>C</i>
$\{A,D\}$	10	
{ <i>B</i> , <i>C</i> }	11	B, C
{ <i>B</i> , <i>D</i> }	9	
$\{C,D\}$	4	
$\{A,B,C\}$	20	(none)
$\{A,B,D\}$	18	AB
$\{A,C,D\}$	13	AC
$\{B,C,D\}$	12	BC
$\{A, B, C, D\}$	21	(none)

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Definitions

Definition (Critical Count)

The <u>critical count</u> of a player is the number of possible coalitions in which he is a critical player.

Definition (Banzhaf Power Index)

The Banzhaf power index (BPI) of a player is that player's critical count divided by the total of all players' critical counts.

Definition (Banzhaf Power Distribution)

The Banzhaf power distribution is the set of BPI's for all the players.

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Example

• Find the power distribution in [14:9,8,3,1].

Example

- Find the power distribution in [14 : 9, 8, 3, 1].
- Does this sound right?

Example

• Find the power distribution in [11:9,8,3,1].

Example

- Find the power distribution in [11:9,8,3,1].
- Does this sound right?

Example

- Use the Javascript program to find the Banzhaf Power Indexes in the following situations.
- [14:6,5,5,4].
- [15:6,5,5,4].
- [16:6,5,5,4].
- [17:6,5,5,4].

Example (Stolen from Wikipedia)

- California has 55 electoral votes, Texas as 34, and New York as 31.
- Total = 120.
- If those were the only three states, then we would have [61:55,34,31].
- Find the power distribution.

Example (Also stolen from Wikipedia)

- Replace New York with Ohio, with 20 electoral votes.
- Total = 109.
- The situation now is [55: 55, 34, 20].
- How has the power distribution changed?

Example

- Find the power distribution in [9:5,4,3,2,1].
- You are *E* and you would like to buy one vote from another player. From which player should you buy it?

Example

- Consider the situation [*q* : 3, 3, 2, 1].
- What quota(s) q makes the power distribution most balanced?
- What quota(s) q makes the power distribution most unbalanced?

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 Chapter 2: Exercises 11, 12, 13, 14, 15, 17, 19; 69, 71. (You may want to use the Javascript program for 69 and 71.)