

# Banzhaf Power

## Lecture 13 Section 2.2

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- 1 Coalitions
- 2 Critical Players
- 3 The Banzhaf Power Index
- 4 Examples
- 5 Assignment

# Outline

- 1 Coalitions
- 2 Critical Players
- 3 The Banzhaf Power Index
- 4 Examples
- 5 Assignment

# Coalitions

## Definition (Coalition)

A **coalition** is a group of players who agree to vote as a block. A **winning coalition** is a coalition whose votes add up to at least the quota. A **losing coalition** is a coalition whose votes add up to less than the quota.

# Coalitions

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- 5 players?
- In general, if there are  $n$  players, then there are  $2^n - 1$  possible coalitions.



# Coalitions

- If there are 3 players, how many possible coalitions are there?
- How about 4 players?
- 5 players?
- In general, if there are  $n$  players, then there are  $2^n - 1$  possible coalitions.
- If we count the empty set (the coalition with nobody in it), then there are  $2^n$ .

# Listing Coalitions

- The best way to list the possible coalitions is by size.
  - Start with the empty set (or skip it).
  - Consider all coalitions of a single player:  $A$ ,  $B$ ,  $C$ , ...
  - Then consider all coalitions of two players by adding a player to the singleton coalitions:  $AB$ ,  $AC$ ,  $BC$ , ...
  - Then coalitions of three players, then four players, and so on.

# Listing Coalitions By Size

By Size

$\emptyset$

The null set

# Listing Coalitions By Size

By Size

$\emptyset$

A

B

C

Coalitions of size 1

# Listing Coalitions By Size

By Size

|             |   |    |
|-------------|---|----|
| $\emptyset$ | A | AB |
|             | B | AC |
|             | C | BC |

Coalitions of size 2

# Listing Coalitions By Size

By Size

|             |   |    |     |
|-------------|---|----|-----|
|             | A | AB |     |
| $\emptyset$ | B | AC | ABC |
|             | C | BC |     |

Coalition of size 3

# Listing Coalitions By “Choice”

- Half of the coalitions include  $A$  and half do not include  $A$ .
- Half of those that include  $A$  also include  $B$  and half do not. The same for the half that do not include  $A$ .
- Etc.

# Outline

1 Coalitions

**2 Critical Players**

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# Critical Players

## Definition (Critical Player)

A **critical player** of a coalition is a player whose membership in that coalition takes it from a losing coalition to a winning coalition. That is, it is a winning coalition, but if he left it, then it would be a losing coalition.

# An Example

## Example (Coalitions)

Consider the voting system  $[4 : 3, 2, 1]$ . Make a table of all possible coalitions and their critical players.

| Coalition     | Weight | Critical Players |
|---------------|--------|------------------|
| $\{A\}$       |        |                  |
| $\{B\}$       |        |                  |
| $\{C\}$       |        |                  |
| $\{A, B\}$    |        |                  |
| $\{A, C\}$    |        |                  |
| $\{B, C\}$    |        |                  |
| $\{A, B, C\}$ |        |                  |

# An Example

## Example (Coalitions)

Consider the voting system  $[4 : 3, 2, 1]$ . Make a table of all possible coalitions and their critical players.

| Coalition     | Weight | Critical Players |
|---------------|--------|------------------|
| $\{A\}$       | 3      |                  |
| $\{B\}$       | 2      |                  |
| $\{C\}$       | 1      |                  |
| $\{A, B\}$    | 5      |                  |
| $\{A, C\}$    | 4      |                  |
| $\{B, C\}$    | 3      |                  |
| $\{A, B, C\}$ | 6      |                  |

# An Example

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Consider the voting system  $[4 : 3, 2, 1]$ . Make a table of all possible coalitions and their critical players.

| Coalition     | Weight | Critical Players |
|---------------|--------|------------------|
| $\{A\}$       | 3      |                  |
| $\{B\}$       | 2      |                  |
| $\{C\}$       | 1      |                  |
| $\{A, B\}$    | 5      | $A, B$           |
| $\{A, C\}$    | 4      | $A, C$           |
| $\{B, C\}$    | 3      |                  |
| $\{A, B, C\}$ | 6      | $A$              |

# An Example

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Consider the voting system  $[4 : 3, 2, 1]$ . Make a table of all possible coalitions and their critical players.

| Coalition     | Weight | Critical Players |
|---------------|--------|------------------|
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| $\{B\}$       | 2      |                  |
| $\{C\}$       | 1      |                  |
| $\{A, B\}$    | 5      | $A, B$           |
| $\{A, C\}$    | 4      | $A, C$           |
| $\{B, C\}$    | 3      |                  |
| $\{A, B, C\}$ | 6      | $A$              |

Notice that  $A$  has veto power, but  $A$  is not a dictator.

# Another Example

## Example (Coalitions)

What if the quota were lowered to 3?

| Coalition     | Weight | Critical Players |
|---------------|--------|------------------|
| $\{A\}$       |        |                  |
| $\{B\}$       |        |                  |
| $\{C\}$       |        |                  |
| $\{A, B\}$    |        |                  |
| $\{A, C\}$    |        |                  |
| $\{B, C\}$    |        |                  |
| $\{A, B, C\}$ |        |                  |

# Another Example

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What if the quota were lowered to 3?

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| $\{C\}$       | 1      |                  |
| $\{A, B\}$    | 5      |                  |
| $\{A, C\}$    | 4      |                  |
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| $\{A, B, C\}$ | 6      |                  |

# Another Example

## Example (Coalitions)

What if the quota were lowered to 3?

| Coalition     | Weight | Critical Players |
|---------------|--------|------------------|
| $\{A\}$       | 3      | $A$              |
| $\{B\}$       | 2      |                  |
| $\{C\}$       | 1      |                  |
| $\{A, B\}$    | 5      | $A$              |
| $\{A, C\}$    | 4      | $A$              |
| $\{B, C\}$    | 3      | $B, C$           |
| $\{A, B, C\}$ | 6      |                  |



# Another Example

## Example (Coalitions)

What if the quota were lowered to 3?

| Coalition     | Weight | Critical Players |
|---------------|--------|------------------|
| $\{A\}$       | 3      | $A$              |
| $\{B\}$       | 2      |                  |
| $\{C\}$       | 1      |                  |
| $\{A, B\}$    | 5      | $A$              |
| $\{A, C\}$    | 4      | $A$              |
| $\{B, C\}$    | 3      | $B, C$           |
| $\{A, B, C\}$ | 6      |                  |

Notice that  $A$  is a dictator, but  $A$  does not have veto power.

# An Example

## Example (Coalitions)

Consider the voting system  $[11 : 9, 8, 3, 1]$ .

| Coalition        | Weight | Critical Players |
|------------------|--------|------------------|
| $\{A\}$          |        |                  |
| $\{B\}$          |        |                  |
| $\{C\}$          |        |                  |
| $\{D\}$          |        |                  |
| $\{A, B\}$       |        |                  |
| $\{A, C\}$       |        |                  |
| $\{A, D\}$       |        |                  |
| $\{B, C\}$       |        |                  |
| $\{B, D\}$       |        |                  |
| $\{C, D\}$       |        |                  |
| $\{A, B, C\}$    |        |                  |
| $\{A, B, D\}$    |        |                  |
| $\{A, C, D\}$    |        |                  |
| $\{B, C, D\}$    |        |                  |
| $\{A, B, C, D\}$ |        |                  |

# An Example

## Example (Coalitions)

Consider the voting system  $[11 : 9, 8, 3, 1]$ .

| Coalition        | Weight | Critical Players |
|------------------|--------|------------------|
| $\{A\}$          | 9      |                  |
| $\{B\}$          | 8      |                  |
| $\{C\}$          | 3      |                  |
| $\{D\}$          | 1      |                  |
| $\{A, B\}$       | 17     |                  |
| $\{A, C\}$       | 12     |                  |
| $\{A, D\}$       | 10     |                  |
| $\{B, C\}$       | 11     |                  |
| $\{B, D\}$       | 9      |                  |
| $\{C, D\}$       | 4      |                  |
| $\{A, B, C\}$    | 20     |                  |
| $\{A, B, D\}$    | 18     |                  |
| $\{A, C, D\}$    | 13     |                  |
| $\{B, C, D\}$    | 12     |                  |
| $\{A, B, C, D\}$ | 21     |                  |

# An Example

## Example (Coalitions)

Consider the voting system  $[11 : 9, 8, 3, 1]$ .

| Coalition        | Weight | Critical Players |
|------------------|--------|------------------|
| $\{A\}$          | 9      |                  |
| $\{B\}$          | 8      |                  |
| $\{C\}$          | 3      |                  |
| $\{D\}$          | 1      |                  |
| $\{A, B\}$       | 17     | $A, B$           |
| $\{A, C\}$       | 12     | $A, C$           |
| $\{A, D\}$       | 10     |                  |
| $\{B, C\}$       | 11     | $B, C$           |
| $\{B, D\}$       | 9      |                  |
| $\{C, D\}$       | 4      |                  |
| $\{A, B, C\}$    | 20     | (none)           |
| $\{A, B, D\}$    | 18     | $AB$             |
| $\{A, C, D\}$    | 13     | $AC$             |
| $\{B, C, D\}$    | 12     | $BC$             |
| $\{A, B, C, D\}$ | 21     | (none)           |

# An Example

## Example (Coalitions)

Consider the voting system  $[11 : 9, 8, 3, 1]$ .

| Coalition        | Weight | Critical Players |
|------------------|--------|------------------|
| $\{A\}$          | 9      |                  |
| $\{B\}$          | 8      |                  |
| $\{C\}$          | 3      |                  |
| $\{D\}$          | 1      |                  |
| $\{A, B\}$       | 17     | $A, B$           |
| $\{A, C\}$       | 12     | $A, C$           |
| $\{A, D\}$       | 10     |                  |
| $\{B, C\}$       | 11     | $B, C$           |
| $\{B, D\}$       | 9      |                  |
| $\{C, D\}$       | 4      |                  |
| $\{A, B, C\}$    | 20     | (none)           |
| $\{A, B, D\}$    | 18     | $AB$             |
| $\{A, C, D\}$    | 13     | $AC$             |
| $\{B, C, D\}$    | 12     | $BC$             |
| $\{A, B, C, D\}$ | 21     | (none)           |

Notice that  $A$  has veto power but  $A$  is not a dictator

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# Definitions

## Definition (Critical Count)

The **critical count** of a player is the number of possible coalitions in which he is a critical player.

## Definition (Banzhaf Power Index)

The **Banzhaf power index (BPI)** of a player is that player's critical count divided by the total of all players' critical counts.

## Definition (Banzhaf Power Distribution)

The **Banzhaf power distribution** is the set of BPI's for all the players.

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# Example

## Example

- Find the power distribution in  $[14 : 9, 8, 3, 1]$ .

# Example

## Example

- Find the power distribution in  $[14 : 9, 8, 3, 1]$ .
- Does this sound right?

# Example

## Example

- Find the power distribution in  $[11 : 9, 8, 3, 1]$ .

# Example

## Example

- Find the power distribution in  $[11 : 9, 8, 3, 1]$ .
- Does this sound right?

# Example

## Example

- Use the Javascript program to find the Banzhaf Power Indexes in the following situations.
- $[14 : 6, 5, 5, 4]$ .
- $[15 : 6, 5, 5, 4]$ .
- $[16 : 6, 5, 5, 4]$ .
- $[17 : 6, 5, 5, 4]$ .

# Example

## Example (Stolen from Wikipedia)

- California has 55 electoral votes, Texas as 34, and New York as 31.
- Total = 120.
- If those were the only three states, then we would have  $[61 : 55, 34, 31]$ .
- Find the power distribution.

# Example

## Example (Also stolen from Wikipedia)

- Replace New York with Ohio, with 20 electoral votes.
- Total = 109.
- The situation now is  $[55 : 55, 34, 20]$ .
- How has the power distribution changed?

# Example

## Example

- Find the power distribution in  $[9 : 5, 4, 3, 2, 1]$ .
- You are  $E$  and you would like to buy one vote from another player. From which player should you buy it?



# Example

## Example

- Consider the situation  $[q : 3, 3, 2, 1]$ .
- What quota(s)  $q$  makes the power distribution most balanced?
- What quota(s)  $q$  makes the power distribution most unbalanced?

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# Assignment

## Assignment

- Chapter 2: Exercises 11, 12, 13, 14, 15, 17, 19; 69, 71. (You may want to use the Javascript program for 69 and 71.)